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On System of Generalized Vector Quasi-Equilibrium Problems with Set-Valued Maps

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Abstract. In this paper, we introduce four new types of the system of generalized vector quasi-equilibrium problems with set-valued maps which include system of vector quasi-equilibrium problems, system of vector equilibrium problems, system of variational inequality problems, and vector equilibrium problems in the literature as special cases. We prove the existence of solutions for such kinds of system of generalized vector quasi-equilibrium problems. Consequently, we derive some existence results of a solution for the system of vector quasi-equilibrium problems and the generalized Debreu type equilibrium problem for vector-valued functions.

Key words: C_{i-x}-0-partially diagonally quasiconvex, generalized Debreu type equilibrium problem, maximal element theorem, Φ-condensing map, system of generalized vector quasi-equilibrium problems, system of vector quasi-equilibrium problems

1. Introduction and Formulations

In the recent years, the equilibrium problems with vector-valued functions and set-valued maps have been studied in [1–8] and the references therein. Very recently, Hou et al. [9] introduced a class of generalized vector quasi-equilibrium problems which includes the models in [1–8] and the vector quasi-equilibrium problems in [10–13] as special cases. They established some existence results of a solution for the generalized vector quasi-equilibrium problems. The system of vector quasi-equilibrium problems, i.e., a family of quasi-equilibrium problems for vector-valued bifunctions defined on a product set, was introduced by Ansari et al. [14] with applications in Debreu type equilibrium problem for vector-valued functions. As generalizations of the above models, we introduce some new types of system of generalized vector quasi-equilibrium problems, i.e., a family of quasi-equilibrium problems for set-valued maps defined on a product set.

Throughout this paper, for a set A in a topological space, we denote by coA, $intA$, $\overline{co}A$ the convex hull, interior, and the convex closure of A, respectively.

Let I be an index set. For each $i \in I$, let Z_i, E_i and F_i be topological vector spaces. Consider two family of nonempty convex subsets $\{X_i\}_{i\in I}$ with $X_i \subseteq E_i$ and $\{Y_i\}_{i\in I}$ with $Y_i \subseteq F_i$. Let

$$E = \prod_{i\in I} E_i, \quad X = \prod_{i\in I} X_i, \quad F = \prod_{i\in I} F_i \text{ and } Y = \prod_{i\in I} Y_i$$

An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i, therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$. Similarly, an element of the set Y will be denoted by $y = (y^i, y_i) \in Y^i \times Y_i$. For each $i \in I$, let $C_i : X \to 2^{Z_i}$, $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty values, and let $\Psi_i : X \times Y \times X_i \to 2^{Z_i}$ be a set-valued map.
The following classes of system of generalized vector quasi-equilibrium problems with set-valued maps are of interest to us:

(I) Weak type I system of generalized vector quasi-equilibrium problems (in short, WI-SGVQEP): Find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x}, \bar{y}, z_i) \subseteq Z_i \setminus (-intC_i(\bar{x})),$$
$$\forall z_i \in D_i(\bar{x}).$$

(II) Weak type II system of generalized vector quasi-equilibrium problems (in short, WII-SGVQEP): Find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x}, \bar{y}, z_i) \not\subseteq -intC_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

(III) Strong type I system of generalized vector quasi-equilibrium problems (in short, SI-SGVQEP): Find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x}, \bar{y}, z_i) \subseteq C_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

(IV) Strong type II system of generalized vector quasi-equilibrium problems (in short, SII-SGVQEP): Find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x}, \bar{y}, z_i) \cap C_i(\bar{x}) \neq \emptyset, \quad \forall z_i \in D_i(\bar{x}).$$

Remark 1.1. In Problems (I) and (II), it is assumed that $intC_i(x)\neq\emptyset$ for each $i\in I$ and for all $x\in X$.

PROPOSITION 1.1. (1) *Assume that* $\forall i\in I$ *and* $\forall x\in X$, *the set* $C_i(x)$ *is nonempty, then* $(\bar{x},\bar{y})\in X\times Y$ *solves (SI-SGVQEP) implies* $(\bar{x},\bar{y})$ *solves (SII-SGVQEP);*

(2) *Assume that* $\forall i\in I$ *and* $\forall x\in X$, *the set* $C_i(x)$ *is a proper closed convex cone with apex at the origin with* $intC_i(x)\neq\emptyset$. *Then the following statements hold:*

(a) $(\bar{x},\bar{y})\in X\times Y$ *solves (WI-SGVQEP) implies* $(\bar{x},\bar{y})$ *solves (WII-SGVQEP);*
(b) $(\bar{x},\bar{y})\in X\times Y$ *solves (SI-SGVQEP) implies* $(\bar{x},\bar{y})$ *solves (WI-SGVQEP);*
(c) $(\bar{x},\bar{y})\in X\times Y$ *solves (SII-SGVQEP) implies* $(\bar{x},\bar{y})$ *solves (WII-SGVQEP).*

It is worth noting that the above four kinds of system of generalized vector quasi-equilibrium problems encompass almost all of the system of vector quasi-equilibrium problems, system of vector equilibrium problems, system of vector variational inequality problems and vector equilibrium problems in the literature. See the examples below.

(a) If the index set I is singleton, then Problems I–IV reduce to find $(\bar{x},\bar{y})$ in $X\times Y$ such that $\bar{x}\in D(\bar{x})$, $\bar{y}\in T(\bar{x})$, and for all $z\in D(\bar{x})$,

$$\Psi(\bar{x},\bar{y},z)\subseteq Z\backslash(-intC(\bar{x})),\quad \Psi(\bar{x},\bar{y},z)\not\subseteq(-intC(\bar{x})),$$
$$\Psi(\bar{x},\bar{y},z)\subseteq C(\bar{x})\quad\text{or}\quad \Psi(\bar{x},\bar{y},z)\cap C(\bar{x})\neq\emptyset,$$

respectively. These generalized vector quasi-equilibrium problems were introduced and studied by Hou et al. [9].

(b) If the set-valued map Ψ_i is replaced by a single-valued function $f_i: X\times Y\times X_i\to Z_i$ for each $i\in I$, then both the (WI-SGVQEP) and the (WII-SGVQEP) reduce to the weak type system of vector quasi-equilibrium problems (in Short, W-SVQEP), which is to find $(\bar{x},\bar{y})=(\bar{x}^i,\bar{x}_i,\bar{y}^i,\bar{y}_i)$ in $X\times Y$ such that for each $i\in I$,

$$\bar{x}_i\in D_i(\bar{x})\quad\text{and}\quad \bar{y}_i\in T_i(\bar{x}):\ f_i(\bar{x},\bar{y},z_i)\notin -intC_i(\bar{x}),\quad \forall z_i\in D_i(\bar{x}).$$

Both the (SI-SGVQEP) and the (SII-SGVQEP) reduce to the strong type system of vector quasi-equilibrium problems (in Short, S-SVQEP), which is to find $(\bar{x},\bar{y})=(\bar{x}^i,\bar{x}_i,\bar{y}^i,\bar{y}_i)$ in $X\times Y$ such that for each $i\in I$,

$$\bar{x}_i\in D_i(\bar{x})\quad\text{and}\quad \bar{y}_i\in T_i(\bar{x}):\ f_i(\bar{x},\bar{y},z_i)\in C_i(\bar{x}),\quad \forall z_i\in D_i(\bar{x}).$$

For each $i \in I$, let $\phi_i : X \times Y \to Z_i$ be a vector-valued function. And we define a trifunction $f_i : X \times Y \times X_i \to Z_i$ as $f_i(x, y, u_i) = \phi_i(x^i, y, u_i) - \phi_i(x, y)$, $\forall (x, y, u_i) \in X \times Y \times X_i$. Then the (W-SVQEP) reduces to the generalized Debreu type equilibrium problem for vector-valued functions (in short, G-Debreu VEP), which is to find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \quad \text{and} \quad \bar{y}_i \in T_i(\bar{x}) : \phi_i(\bar{x}^i, \bar{y}, z_i) - \phi_i(\bar{x}, \bar{y}) \notin -intC_i(\bar{x}), \qquad \forall\ z_i \in D_i(\bar{x}).$$

We denote by R and R^+ the set of real numbers and the set of real non-negative numbers, respectively. For each $i \in I$, if $Z_i = R$ and $C_i(x) = R^+$ for all $x \in X$, then both the (W-SVQEP) and the (S-SVQEP) reduce to the system of quasi-equilibrium problems (in short, SQEP), which is to find $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \quad \text{and} \quad \bar{y}_i \in T_i(\bar{x}) : \ f_i(\bar{x}, \bar{y}, z_i) \geqslant 0, \forall z_i \in D_i(\bar{x}).$$

Let $Y = \{\bar{y}\}$ and for each $i \in I$, $T_i(x) = \{\bar{y}_i\}$ for all $x \in X$, let $\varphi_i : X \times X_i \to Z_i$ and $h_i : X \times Y \to Z_i$, respectively, be defined as $\varphi_i(x, z_i) = f_i(x, \bar{y}, z_i)$, $\forall (x, z_i) \in X \times X_i$ and $h_i(x) = \phi_i(x, \bar{y})$, $\forall x \in X$, then the (W-SVQEP) and the (G-Debreu VEP), respectively, reduce to the system of vector quasi-equilibrium problems and the (Debreu VEP) introduced by Ansari et al. [14]. And the (SQEP) reduces to the mathematical model in [15, p. 286] and [16, pp. 152–153].

(c) Let $Y = \{\bar{y}\}$. For each $i \in I$ and for all $x \in X$, let $T_i(x) = \{\bar{y}_i\}$. Let $F_i : X \times X_i \to 2^{Z_i}$ be defined as $F_i(x, z_i) = \Psi_i(x, \bar{y}, z_i)$, then the (WII-SGVQEP) reduces to find $\bar{x}$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) : F_i(\bar{x}, z_i) \not\subseteq -intC_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

This was researched by Ansari and Khan [17] and contains as special cases the system of vector equilibrium problems with set-valued maps in [18], the system of vector equilibrium problems in [19], and the system of variational inequalities in [20–24].

The rest of this paper is arranged in the following manner. The next section deals with some preliminary definitions, notations and results which will be used in the sequel. In Section 3, we establish existence results for a solution to the (WI-SGVQEP), the (WII-SGVQEP), the (SI-SGVQEP) and the (SII-SGVQEP) with or without involving Φ-condensing maps by using the same techniques in [14]. Consequently, we derive some existence results of a solution for the weak type system of vector quasi-equilibrium problems. In Section 4, as applications of the results of Section 3, we derive some existence results of a solution for the (G-Debreu VEP).

2. Preliminaries

In order to prove the main results, we need the following definitions.

DEFINITION 2.1. For each $i \in I$, let $C_i : X \to 2^{Z_i}$ and $\Psi : X \times X_i \to 2^{Z_i}$ be set-valued maps, $\varphi : X \times X_i \to Z_i$ a vector-valued function. Then

(i) Ψ is called to be weak type I C_{i-x}-0-partially diagonally quasiconvex (WIC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\Psi(x, z_{i_j}) \subseteq Z_i \backslash (-intC_i(x))$.

(ii) Ψ is called to be weak type II C_{i-x}-0-partially diagonally quasiconvex (WIIC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\Psi(x, z_{i_j}) \nsubseteq -intC_i(x)$.

(iii) Ψ is called to be strong type I C_{i-x}-0-partially diagonally quasiconvex (SIC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\Psi(x, z_{i_j}) \subseteq C_i(x)$.

(iv) Ψ is called to be strong type II C_{i-x}-0-partially diagonally quasiconvex (SIIC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\Psi(x, z_{i_j}) \cap C_i(x) \neq \emptyset$.

(v) φ is called to be weak type C_{i-x}-0-partially diagonally quasiconvex (WC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\varphi(x, z_{i_j}) \notin -intC_i(x)$.

(vi) φ is called to be strong type C_{i-x}-0-partially diagonally quasiconvex (SC-PDQC, in short) in the second argument if, for any finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \subseteq X_i$, and for all $x \in X$ with $x_i \in co\wedge_i$, there exists $j \in \{1, 2, \dots, n\}$ such that $\varphi(x, z_{i_j}) \in C_i(x)$.

Remark 2.1. It is assumed that $intC_i(x) \neq \emptyset$ for each $i \in I$ and for all $x \in X$ in cases (i), (ii) and (v), and that $C_i(x) \neq \emptyset$ for each $i \in I$ and for all $x \in X$ in cases (iii), (iv) and (vi).

Remark 2.2. If the index set I is a singleton, then the (I), (II), (III) and (IV) in Definition 2.1, respectively, reduce to the weak type I C-diagonal quasiconvexity, the weak type II C-diagonal quasiconvexity, the strong type I C-diagonal quasiconvexity, the strong type II C-diagonal quasiconvexity of Ψ (see Definition 2.3 in [9] and Definition 1 in [8]).

Remark 2.3. For each $i \in I$, if $Z_i = R$ and $C_i(x) = R^+$ for all $x \in X$, then both the weak type C_{i-x}-0-partially diagonal quasiconvexity and the strong type C_{i-x}-0-partially diagonal quasiconvexity of φ_i reduce to the 0-partially diagonal quasiconvexity (i.e., [25, Definition 3]), which in turn reduces to the γ-diagonal quasiconvexity in [26] if $I = \{1\}$, here $\gamma = 0$.

PROPOSITION 2.1. (1) *Assume that* $\forall i \in I$ *and* $\forall x \in X$, *the set* $C_i(x)$ *is nonempty, then SIC-PDQC implies SIIC-PDQC;*

(2) *Assume that* $\forall i \in I$ *and* $\forall x \in X$, *the set* $C_i(x)$ *is a proper closed convex cone with apex at the origin with* $intC_i(x) \neq \emptyset$. *Then,*

(a) *SIC-PDQC implies WIC-PDQC;*
(b) *WIC-PDQC implies WIIC-PDQC;*
(c) *SIIC-PDQC implies WIIC-PDQC;*
(d) *SC-PDQC implies WC-PDQC.*

DEFINITION 2.2. [14, 27]. Let M be a nonempty convex subset of a topological vector space E and Z a real topological space with a closed and convex cone P with apex at the origin. A vector-valued function $\varphi : M \to Z$ is called

(i) P-quasifunction iff, for all $z \in Z$, the set $\{x \in M : \varphi(x) \in z - P\}$ is convex.
(ii) natural P-quasifunction iff, $\forall x, y \in M$, and $\lambda \in [0, 1]$, $\varphi(\lambda x + (1 - \lambda) y) \in co\{\varphi(x), \varphi(y)\} - P$.

DEFINITION 2.3. [3]. Let $C : X \to 2^Z$ be a set-valued map with nonempty values. Then the set-valued map $\Psi : X \times X \to 2^Z$ is called to be $C(x)$-quasiconvex-like if, for all $x \in X$, $y_1, y_2 \in X$, and $\alpha \in [0, 1]$, we have either

$$\Psi(x, \alpha y_1 + (1 - \alpha) y_2) \subseteq \Psi(x, y_1) - C(x)$$

or

$$\Psi(x, \alpha y_1 + (1 - \alpha) y_2) \subseteq \Psi(x, y_2) - C(x).$$

The following example shows that there exists a set-valued map $\Psi : X \times X_i \to 2^{Z_i}$ which is (SIC-PDQC) in the second argument. By proposition 2.1, we know that Ψ is also (SIIC-PDQC), (WIC-PDQC) and (WIIC-PDQC) in the second argument. However, Ψ may not be $C_i(x)$-quasiconvex-like.

EXAMPLE 2.1. Let I be any finite index set. For each $i \in I$, let E_i be a real normed space with dual space $E_i{}^*$, $X_i \subset E_i$, $Z_i = R$, $C_i : X \to 2^{Z_i}$ be defined as $C_i(x) = C_i = R^+, \forall x \in X$. Let $\| \bullet \|_i$ denote the norm on E_i. If we define a norm on E as follows

$$\|x\| = \sum_{i=1}^{n} \|x_i\|_i, \quad \forall x = (x_1, x_2, \ldots, x_n) \in E,$$

then it is easy to verify that $\| \bullet \|$ is a norm on E. And hence E is also a real normed space. Let $[e_1, e_2]$ denote the line segment joining e_1 and e_2. Choose $p_i \in {E_i}^*$, we define a set-valued map $\Psi : X \times X_i \to 2^{Z_i}$ as

$$\Psi(x, z_i) = \{\langle u, z_i - x_i \rangle : u \in [\|x\| \|z_i\|_i p_i, 2\|x\| \|z_i\|_i p_i]\}, \quad \forall (x, z_i) \in X \times X_i,$$

Then, Ψ is (SIC-PDQC) in the second argument. Otherwise, there exists finite set $\wedge_i = \{z_{i_1}, z_{i_2}, \ldots, z_{i_n}\} \subseteq X_i$, and there is $x \in X$ with $x_i = \sum_{j=1}^{n} \alpha_j z_{i_j}$ ($\alpha_j \geqslant 0, \sum_{j=1}^{n} \alpha_j = 1$) such that for all $j = 1, 2, \ldots, n$, $\Psi(x, z_{i_j}) \not\subseteq C_i(x)$. Then for each j, there exists $\bar{\lambda}_j \in [0, 1]$ such that

$$\langle \bar{\lambda}_j \|x\| \|z_{i_j}\|_i p_i + (1 - \bar{\lambda}_j) 2\|x\| \|z_{i_j}\|_i p_i, z_{i_j} - x_i \rangle < 0,$$

It follows that

$$\langle p_i, z_{i_j} - x_i \rangle < 0, \quad j = 1, 2, \ldots, n.$$

Then we have

$$0 > \sum_{j=1}^{n} \alpha_j \langle p_i, z_{i_j} - x_i \rangle = \langle p_i, x_i - x_i \rangle = 0,$$

a contradiction. So $\Psi(x, y_i)$ is (SIC-PDQC) in z_i.

However, $\Psi(x, z_i)$ is not $C_i(x)$-quasiconvex-like. In fact, choose $\hat{x} \in E$ such that $\langle p_i, \hat{x}_i \rangle > 0$, set $z_{i_1} = \frac{1}{2}\hat{x}_i$, $z_{i_2} = -\frac{1}{2}\hat{x}_i$. Then we have

$$\Psi(\hat{x}, z_{i_1}) = \left\{ \alpha \|\hat{x}\| \|\hat{x}_i\|_i \langle p_i, \hat{x}_i \rangle : -\frac{1}{2} \leqslant \alpha \leqslant -\frac{1}{4} \right\} \subseteq -int C_i(\hat{x}).$$
$$\Psi(\hat{x}, z_{i_2}) = \left\{ \alpha \|\hat{x}\| \|\hat{x}_i\|_i \langle p_i, \hat{x}_i \rangle : -\frac{3}{2} \leqslant \alpha \leqslant -\frac{3}{4} \right\} \subseteq -int C_i(\hat{x}).$$

But for $z_{i_0} = \frac{1}{2}(z_{i_1} + z_{i_2}) = 0$, we have

$$\Psi\left(\hat{x}, \frac{1}{2}(z_{i_1} + z_{i_2})\right) = \Psi(\hat{x}, z_{i_0}) = \{0\} \not\subseteq \Psi(\hat{x}, z_{i_j}) - C_i(\hat{x}), \quad j = 1, 2.$$

DEFINITION 2.4. [28]. Let X and Y be two topological spaces, $T : X \to 2^Y$ be a set-valued map. Then T is said to be upper semicontinuous if the set $\{x \in X : T(x) \subseteq V\}$ is open in X for every open subset V of Y. T is said to be lower semicontinuous if the set $\{x \in X : T(x) \cap V\}$ is open in X for every open subset V of Y. T is said to have open lower sections if the set $T^{-1}(y)=\{x\in X:y\in T(x)\}$ is open in X for each $y\in Y$.

DEFINITION 2.5. [29]. Let E be a Hausdorff topological space and L a lattice with least element, denoted by 0. A map $\Phi\colon 2^E\to L$ is a measure of noncompactness provided that the following conditions hold $\forall M,N\in 2^E$:

(i) $\Phi(M)=0$ iff M is precompact (i.e., it is relatively compact).
(ii) $\Phi(\overline{co}M)=\Phi(M)$.
(iii) $\Phi(M\cup N)=\max\{\Phi(M),\Phi(N)\}$.

DEFINITION 2.6. [29]. Let $\Phi:2^E\to L$ be a measure of noncompactness on E and $X\subseteq E$. A set-valued map $T:X\to 2^E$ is called Φ-condensing provided that, if $M\subseteq X$ with $\Phi(T(M))\geqslant\Phi(M)$, then M is relatively compact.

Remark 2.4. Note that every set-valued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If E is locally convex and $T:X\to 2^E$ is a compact set-valued map (i.e., $T(X)$ is precompact), then T is Φ-condensing for any measure of noncompactness Φ. It is clear that if $T:X\to 2^E$ is Φ-condensing and $T^*:X\to 2^E$ satisfies $T^*(x)\subseteq T(x)$ $\forall x\in X$, then T^* is also Φ-condensing.

We shall use the following particular forms of two maximal element theorems for a family of set-valued maps due to Deguire et al. [30, Theorem 7] and Chebbi and Florenzano [31, Corollary 4].

LEMMA 2.1. [14, 18, 30]. *Let $\{X_i\}_{i\in I}$ be a family of nonempty convex subsets where each X_i is contained in a Hausdorff topological vector space E_i. For each $i\in I$, let $S_i:X\to 2^{X_i}$ be a set-valued map such that*

(i) *for each $i\in I$, $S_i(x)$ is convex,*
(ii) *for each $x\in X$, $x_i\notin S_i(x)$,*
(iii) *for each $y_i\in X_i$, $S_i^{-1}(y_i)$ is open in X.*
(iv) *there exist a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i for each $i\in I$ such that for each $x\in X\setminus N$ there exists $i\in I$ satisfying $S_i(x)\cap B_i\neq\emptyset$. Then there exists $\bar{x}\in X$ such that $S_i(\bar{x})=\emptyset$ for all $i\in I$.*

LEMMA 2.2. [14, 31] Let I be any index set and $\{X_i\}_{i\in I}$ be a family of nonempty, closed and convex subsets where each X_i is contained in a locally convex Hausdorff topological vector space E_i. For each $i\in I$, let $S_i:X\to 2^{X_i}$ be a set-valued map. Assume that the set-valued map $S:X\to 2^X$ defined as $S(x)=\prod_{i\in I}S_i(x)$, $\forall x\in X$, is Φ-condensing and the conditions

(i) – (iii) of Lemma 2.1 hold. Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

3. Existence Results

Some existence results of a solution for the four types of system of generalized vector quasi-equilibrium problems without Φ-condensing maps are firstly shown.

THEOREM 3.1. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, E_i and F_i be two Hausdorff topological vector spaces, $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, and the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x) \text{ and } y_i \in T_i(x)\}$ be closed in $X \times Y$. For each $i \in I$, assume that*

(i) $C_i : X \to 2^{Z_i}$ is a set-valued map such that $intC_i(x) \neq \emptyset$ for each $x \in X$;
(ii) $\Psi_i : X \times Y \times X_i \to 2^{Z_i}$ is a set-valued map satisfies:
  (a) $\forall\, z_i \in X_i$, the set $\{(x, y) \in X \times Y : \Psi_i(x, y, z_i) \subseteq -intC_i(x)\}$ is open;
  (b) For each $y \in Y$, $\Psi_i(x, y, z_i)$ is weak type II C_{i-x}-0-partially diagonally quasiconvex in the third argument;
  (c) there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact and convex subsets $B_i \subseteq X_i$, $A_i \subseteq Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \backslash N \times K \exists i \in I$ and $\exists \bar{u}_i \in B_i$, $\bar{v}_i \in A_i$ satisfying $\bar{u}_i \in D_i(x)$, $\bar{v}_i \in T_i(x)$ and $\Psi_i(x, y, \bar{u}_i) \subseteq -intC_i(x)$.

Then, there exists $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x}, \bar{y}, z_i) \nsubseteq -intC_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (WII-SGVQEP) is nonempty.

Proof. For each $i \in I$, let us define a set-valued map $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : \Psi_i(x, y, z_i) \subseteq -intC_i(x)\}, \quad \forall\ (x, y) \in X \times Y.$$

We show first that, for each $i \in I$ and for all $(x, y) = (x^i, x_i, y) \in X \times Y$,

$$x_i \notin coP_i(x, y),. \tag{1}$$

or else, there would exist $i \in I$ and $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x}_i \in co(P_i(\bar{x}, \bar{y}))$. That is, there exist a finite subset $\{z_{i_1}, z_{i_2}, \ldots, z_{i_n}\} \subseteq P_i(\bar{x}, \bar{y})$ such that

$$\bar{x}_i \in co\{z_{i_1}, z_{i_2}, \ldots, z_{i_n}\}.$$

Therefore, we have $\Psi_i(\bar{x}, \bar{y}, z_{i_j}) \subseteq -intC_i(\bar{x})$, $j = 1, 2, \ldots, n$, which contradicts hypothesis (b) of (ii). Therefore, (1) holds.

By condition (a) of (ii), $\forall i \in I$, $\forall\ z_i \in X_i$, the set

$$P_i^{-1}(z_i) = \{(x, y) \in X \times Y : \Psi_i(x, y, z_i) \subseteq -intC_i(x)\}$$

is open in X_i. That is, P_i has open lower sections on $X \times Y$. And by Lemma 2 in [32], we know that $coP_i : X \times Y \to 2^{X_i}$ also has open lower sections. For each $i \in I$, we also define another set-valued map $S_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$S_i(x, y) = \begin{cases} [D_i(x) \cap coP_i(x, y)] \times T_i(x) & \text{if } (x, y) \in W_i, \\ D_i(x) \times T_i(x) & \text{if } (x, y) \notin W_i. \end{cases}$$

Then, it is clear that $\forall i \in I$ and $\forall (x, y) \in X \times Y$, $S_i(x, y)$ is convex, and $(x_i, y_i) \notin S_i(x, y)$. Since $\forall i \in I$ and $\forall (u_i, v_i) \in X_i \times Y_i$,

$$\begin{aligned} S_i^{-1}(u_i, v_i) = & \Big[coP^{-1}(u_i) \cap (D_i^{-1}(u_i) \times Y) \cap (T_i^{-1}(v_i) \times Y)\Big] \\ & \cup \Big[(X \times Y \setminus W_i) \cap (D_i^{-1}(u_i) \times Y) \cap (T_i^{-1}(v_i) \times Y)\Big]. \end{aligned}$$

and $D_i^{-1}(u_i) \times Y$, $T_i^{-1}(v_i) \times Y$, $coP_i^{-1}(u_i)$ and $X \times Y \setminus W_i$ are open in $X \times Y$, we have $S_i^{-1}(u_i, v_i)$ is open in $X \times Y$.

From condition (iv), there exist a nonempty and compact subset $N \times K \subseteq X \times Y$ and a nonempty, compact and convex subset $B_i \times A_i \subseteq X_i \times Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \backslash N \times K \exists i \in I$ and $\exists (\bar{u_i}, \bar{v_i}) \in S_i(x, y) \cap (B_i \times A_i)$. Hence, by Lemma 2.1, $\exists (\bar{x}, \bar{y}) \in X \times Y$ such that $S_i(\bar{x}, \bar{y}) = \emptyset, \forall i \in I$. Since $\forall i \in I$ and $\forall (x, y) \in X \times X$, $D_i(x)$ and $T_i(y)$ are nonempty, we have $(\bar{x}, \bar{y}) \in W_i$ and $D_i(\bar{x}) \cap coP_i(\bar{x}, \bar{y}) = \emptyset$, $\forall i \in I$. This implies $(\bar{x}, \bar{y}) \in W_i$ and $D_i(\bar{x}) \cap P_i(\bar{x}, \bar{y}) = \emptyset$, $\forall i \in I$. Therefore, $\forall i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \bar{y}_i \in T_i(\bar{x}) \quad \text{and} \quad \Psi_i(\bar{x}, \bar{y}, z_i) \not\subseteq -intC_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (WII-SGVQEP) is nonempty. □

Remark 3.1. The condition (b) of (ii) in Theorem 3.1 can be replaced by the following conditions.

(b1) For each $(x, y) \in X \times Y$, the set $P_i(x, y) = \{z_i \in X_i : \Psi_i(x, y, z_i) \subseteq -intC_i(x)\}$ is a convex set;

(b2) For all $x = (x^i, x_i) \in X$, for all $y \in Y$, $\Psi_i(x, y, x_i) \not\subseteq -intC_i(x)$.

In fact, If the condition (b) of (ii) in Theorem 3.1 is not satisfied, then there exist $i \in I$, $y \in Y$, a finite subset $\{z_{i_1}, z_{i_2}, \ldots, z_{i_n}\}$ in X_i, and a point $x=(x^i,x_i)\in X$ with $x_i\in co\{z_{i_1},z_{i_2},\dots,z_{i_n}\}$ such that for each $j=1,2,\dots,n$, $\Psi_i(x,y,z_{i_j})\subseteq -intC_i(x)$, i.e., $z_{i_j}\in P_i(x,y)$. Since $P_i(x,y)=\{z_i\in X_i:\Psi_i(x,y,z_i)\subseteq -intC_i(x)\}$ is a convex set, $x_i\in P_i(x,y)$, i.e., $\Psi_i(x,y,x_i)\subseteq -intC_i(x)$, which contradicts to the condition (b2).

Remark 3.2. The condition (b) of (ii) in Theorem 3.1 can be replaced by the following conditions.

(b1) For each $y\in Y$, $\Psi_i(x,y,z_i)$ is $C_i(x)$-convex-like;
(b2) For all $x=(x^i,x_i)\in X$, for all $y\in Y$, $\Psi_i(x,y,x_i)\not\subseteq -intC_i(x)$.

In fact, for each $i\in I$, let P_i be same as in Remark 3.1. Then by (b1), for each $i\in I$ and for each $(x,y)\in X\times Y$, the set $P_i(x,y)$ is a convex set (see for example the proof of Theorem 2.1 in [3]).

Remark 3.3. Let $Y=\{\bar{y}\}$. For each $i\in I$ and for all $x\in X$, let $D_i(x)=X_i$, $T_i(x)=\{\bar{y}_i\}$. Let $F_i:X\times X_i\to 2^{Z_i}$ be defined as $F_i(x,z_i)=\Psi_i(x,\bar{y},z_i),\forall(x,z_i)\in X\times Y_i$. And let the condition (b) of (ii) in Theorem 3.1 be replaced by the conditions (b1) and (b2) in Remark 3.1 or Remark 3.2, then by Theorem 3.1, we recover Theorems 2 and 3 in [18]. By Example 2.1, we know that Theorem 3.1 extends and generalizes Theorems 2 and 3 in [18] in several ways.

THEOREM 3.2. *Assume that all the hypotheses of Theorem* 3.1 *are satisfied, except that the condition* (ii) *is replaced by*
(ii)* $\Psi_i:X\times Y\times X_i\to 2^{Z_i}$ *satisfies:*

(a) $\forall\ z_i\in X_i$, *the set* $\{(x,y)\in X\times Y:\Psi_i(x,y,z_i)\cap -intC_i(x)\neq\emptyset\}$ *is open;*
(b) *For each* $y\in Y$, $\Psi_i(x,y,z_i)$ *is weak type I* C_{i-x}-0*-partially diagonally quasiconvex in the third argument;*
(c) *there exist nonempty and compact subsets* $N\subseteq X$ *and* $K\subseteq Y$ *and nonempty, compact and convex subsets* $B_i\subseteq X_i$, $A_i\subseteq Y_i$ *for each* $i\in I$ *such that* $\forall(x,y)=(x^i,x_i,y)\in X\times Y\backslash N\times K$ $\exists i\in I$ *and* $\exists\bar{u}_i\in B_i$, $\bar{v}_i\in A_i$ *satisfying* $\bar{u}_i\in D_i(x)$, $\bar{v}_i\in T_i(x)$ *and* $\Psi_i(x,y,\bar{u}_i)\cap -intC_i(x)\neq\emptyset$.

Then, there exists $(\bar{x},\bar{y})=(\bar{x}^i,\bar{x}_i,\bar{y}^i,\bar{y}_i)$ *in* $X\times Y$ *such that for each* $i\in I$,

$$\bar{x}_i\in D_i(\bar{x})\ \text{and}\ \bar{y}_i\in T_i(\bar{x}):\ \Psi_i(\bar{x},\bar{y},z_i)\subseteq Z_i\backslash(-intC_i(\bar{x})),\quad \forall z_i\in D_i(\bar{x}).$$

That is, the solution set of the (*WI-SGVQEP*) *is nonempty.*

Proof. We proceed as in the proof of Theorem 3.1. But we need to modify the set-valued map $P_i:X\times Y\to 2^{X_i}$ $(\forall i\in I)$ to be

$$P_i(x,y)=\{z_i\in X_i:\Psi_i(x,y,z_i)\cap -intC_i(x)\neq\emptyset\},\quad \forall\ (x,y)\in X\times Y.$$

Then, it is easy to show that $x_i \notin co(P_i(x,y))$ for each $i \in I$ and for all $(x,y) = (x^i, x_i, y) \in X \times Y$ due to the condition (b) of (ii)*. The rest of the proof is similar to that of Theorem 3.1. This completes the proof of the theorem. □

THEOREM 3.3. *Assume that all the hypotheses of Theorem 3.1 are satisfied, except that the condition (ii) is replaced by*
*(ii)** $\Psi_i : X \times Y \times X_i \to 2^{Z_i}$ *satisfies:*

(a) $\forall\ z_i \in X_i$, *the set* $\{(x,y) \in X \times Y : \Psi_i(x,y,z_i) \not\subseteq C_i(x)\}$ *is open.*
(b) For each $y \in Y$, $\Psi_i(x,y,z_i)$ *is strong type I* C_{i-x}*-0-partially diagonally quasiconvex in the third argument;*
(c) there exist nonempty and compact subsets $N \subseteq X$ *and* $K \subseteq Y$ *and nonempty, compact and convex subsets* $B_i \subseteq X_i$, $A_i \subseteq Y_i$ *for each* $i \in I$ *such that* $\forall (x,y) = (x^i, x_i, y) \in X \times Y \backslash N \times K \exists i \in I$ *and* $\exists \bar{u}_i \in B_i$, $\bar{v}_i \in A_i$ *satisfying* $\bar{u}_i \in D_i(x)$, $\bar{v}_i \in T_i(x)$ *and* $\Psi_i(x,y,\bar{u}_i) \not\subseteq C_i(x)$.

Then, there exists $(\bar{x},\bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ *in* $X \times Y$ *such that for each* $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \Psi_i(\bar{x},\bar{y},z_i) \subseteq C_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (SI-SGVQEP) is nonempty.

THEOREM 3.4. *Assume that all the hypotheses of Theorem* 3.1 *are satisfied, except that the condition* (ii) *is replaced by*
(ii)* $\Psi_i : X \times Y \times X_i \to 2^{Z_i}$ *satisfies:*

(a) $\forall\ z_i \in X_i$, *the set* $\{(x,y) \in X \times Y : \Psi_i(x,y,z_i) \cap C_i(x) = \emptyset\}$ *is open;*
(b) *For each* $y \in Y$, $\Psi_i(x,y,z_i)$ *is strong type II* C_{i-x}*-0-partially diagonally quasiconvex in the third argument;*
(c) *there exist nonempty and compact subsets* $N \subseteq X$ *and* $K \subseteq Y$ *and nonempty, compact and convex subsets* $B_i \subseteq X_i$, $A_i \subseteq Y_i$ *for each* $i \in I$ *such that* $\forall (x,y) = (x^i, x_i, y) \in X \times Y \backslash N \times K\ \exists i \in I$ *and* $\exists \bar{u}_i \in B_i$, $\bar{v}_i \in A_i$ *satisfying* $\bar{u}_i \in D_i(x)$, $\bar{v}_i \in T_i(x)$ *and* $\Psi_i(x,y,\bar{u}_i) \cap C_i(x) = \emptyset$.

Then, there exists $(\bar{x},\bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ *in* $X \times Y$ *such that for each* $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \ \Psi_i(\bar{x},\bar{y},z_i) \cap C_i(\bar{x}) \neq \emptyset, \quad \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (*S-II-SGVQEP*) *is nonempty.*

The proof of Theorem 3.3 as well as that of Theorem 3.4 are similar to that of Theorem 3.1 or Theorem 3.2; therefore, they are omitted.

Remark 3.4. The condition (a) of (ii) in Theorem 3.1 is satisfied if the following conditions hold $\forall i \in I$:

(1) $M_i = Z_i \backslash (-intC_i) : X \to 2^{Z_i}$ is upper semicontinuous;
(2) For all $z_i \in X_i$, $(x, y) \mapsto \Psi_i(x, y, z_i)$ is upper semicontinuous on $X \times Y$ with compact values.

In fact, we can prove that $Q_i(z_i) = \{(x, y) \in X \times Y : \Psi_i(x, y, z_i) \nsubseteq -intC_i(x)\}$ is closed for all $z_i \in X_i$. Consider a net $(x_t, y_t) \in Q_i(z_i)$ such that $(x_t, y_t) \to (x, y) \in X \times Y$. Since $(x_t, y_t) \in Q_i(z_i)$, there exists $u_t \in \Psi_i(x_t, y_t, z_i)$ such that $u_t \notin -intC_i(x_t)$. From the upper semicontinuity and compact values of Ψ_i on $X \times Y$ and Proposition 1 in [33], it suffices to find a subset $\{u_{t_j}\}$ which converges to some $u \in \Psi_i(x, y, z_i)$, where $u_{t_j} \in \Psi_i(x_{t_j}, y_{t_j}, z_i)$. Since $(x_{t_j}, u_{t_j}) \to (x, u)$, by Proposition 7 in [34, p. 110] and the upper semicontinuity of M_i, it follows that $u \notin -intC_i(x)$, and hence $(x, y) \in Q_i(z_i)$, $Q_i(z_i)$ is closed.

Remark 3.5. The condition (a) of (ii)* in Theorem 3.2 is satisfied if the following hold $\forall i \in I$:

(1) $M_i = Z_i \backslash (-intC_i) : X \to 2^{Z_i}$ is upper semicontinuous;
(2) For all $z_i \in X_i$, $(x, y) \mapsto \Psi_i(x, y, z_i)$ is lower semicontinuous on $X \times Y$.

In fact, we can prove that $Q_i(z_i) = \{(x, y) \in X \times Y : \Psi_i(x, y, z_i) \subseteq Z_i \backslash (-intC_i(x))\}$ is closed for all $z_i \in X_i$. Consider a net $(x_t, y_t) \in Q_i(z_i)$ such that $(x_t, y_t) \to (x, y) \in X \times Y$. Then for each t, $\Psi_i(x_t, y_t, z_i) \subseteq Z_i \backslash (-intC_i(x_t))$. Since $(x, y) \mapsto \Psi_i(x, y, z_i)$ is lower semicontinuous on $X \times Y$, by (v) of Lemma 2 in [12], for any $w \in \Psi_i(x, y, z_i)$, there exists a net w_t such that $w_t \in \Psi_i(x_t, y_t, z_i)$ and w_t converges to w. By Proposition 7 in [34, p. 110] and the upper semicontinuity of M_i, it follows that $w \notin -intC_i(x)$. And hence $(x, y) \in Q_i(z_i)$ and $Q_i(z_i)$ is closed.

Remark 3.6. The condition (a) of (ii)* in Theorem 3.3 is satisfied if the following hold $\forall i \in I$:

(1) $C_i : X \to 2^{Z_i}$ is upper semicontinuous set-valued map with closed values;
(2) For all $z_i \in X_i$, $(x, y) \mapsto \Psi_i(x, y, z_i)$ is lower semicontinuous on $X \times Y$.

Remark 3.7. The condition (a) of (ii)* in Theorem 3.4 is satisfied if the following conditions hold $\forall i \in I$:

(1) $C_i : X \to 2^{Z_i}$ is upper semicontinuous set-valued map with closed values;
(2) For all $z_i \in X_i$, $(x, y) \mapsto \Psi_i(x, y, z_i)$ is upper semicontinuous on $X \times Y$ with compact values.

Then, an existence result of a solution for the (WII-SGVQEP) with Φ-condensing maps is also presented as follows.

THEOREM 3.5. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, E_i and F_i be two locally convex Hausdorff topological vector spaces, $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty, closed and convex subsets, let $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x) \text{ and } y_i \in T_i(x)\}$ be closed in $X \times Y$. And let the set-valued map $D \times T = \left(\prod_{i\in I} D_i \times \prod_{i\in I} T_i\right) : X \times X \to 2^{X\times Y}$ defined as $(D \times T)(x, y) = \prod_{i\in I} D_i(x) \times \prod_{i\in I} T_i(y)$, $\forall (x, y) \in X \times X$ be Φ-condensing. Assume that the conditions* (i) *and* (ii) *of Theorem* 3.1 *hold. Then, the solution set of the* (*WII-SGVQEP*) *is nonempty.*

Proof. In view of Lemma 2.2 and the proof of Theorem 3.1, it is sufficient to show that the set-valued map $S : X \times Y \to 2^{X\times Y}$ defined as $S(x, y) = \prod_{i\in I} S_i(x, y), \forall (x, y) \in X \times Y$, is Φ-condensing, where S_i's are the same as in the proof of Theorem 3.1. By the definition of S_i, $S_i(x, y) \subseteq D_i(x) \times T_i(x)$ for all $(x, y) \in X \times Y$ and for each $i \in I$, and therefore $S(x, y) \subseteq D(x) \times T(x)$ for all $(x, y) \in X \times Y$. Since $D \times T$ is Φ-condensing, by Remark 2.4, we have S is also Φ-condensing. □

Remark 3.8. By similar argument with that of Theorem 3.5, we can easily obtain the existence results of a solution for the (WI-SGVQEP), the (SI-SGVQEP) and the (SII-SGVQEP) with Φ-condensing maps, and they are omitted.

Remark 3.9. Let $Y = \{\bar{y}\}$. For each $i \in I$, let $T_i(x) = \{\bar{y}_i\}$, $\forall x \in X$, and $F_i : X \times X_i \to 2^{Z_i}$ be defined as $F_i(x, z_i) = \Psi_i(x, \bar{y}, z_i)$, $\forall x \in X, \forall z_i \in X_i$. Let the condition (b) of (ii) in Theorems 3.1 and 3.5 be replaced by the conditions (b1) and (b2) in Remark 3.1, then we recover Theorems 3.1 and 3.2 in [17]. Therefore, Theorems 3.1 and 3.5 extend and generalize the main results in [17] in several ways.

Let Ψ_i be replaced by a vector-valued function f_i, by Theorems 3.1 and 3.5, respectively, we have the following two existence results of a solution for the W-SVQEP.

COROLLARY 3.1. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, E_i and F_i be two Hausdorff topological vector spaces, $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, and the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x) \text{ and } y_i \in T_i(x)\}$ be closed in $X \times Y$. For each $i \in I$, assume that*

(i) *$C_i : X \to 2^{Z_i}$ is a set-valued map such that $intC_i(x) \neq \emptyset$ for each $x \in X$ and the set-valued map $M_i = Z_i \backslash (-intC_i) : X \to 2^{Z_i}$ is upper semicontinuous;*

(ii) *the function* $f_i : X \times Y \times X_i \to Z_i$ *satisfies:*

(a) *For all* $z_i \in X_i$*, the map* $(x, y) \mapsto f_i(x, y, z_i)$ *is continuous on* $X \times Y$*;*

(b) *For each* $y \in Y$*,* $f_i(x, y, z_i)$ *is weak type* C_{i-x}*-0-partially diagonally quasiconvex in the third argument;*

(iii) *there exist nonempty and compact subsets* $N \subseteq X$ *and* $K \subseteq Y$ *and nonempty, compact and convex subsets* $B_i \subseteq X_i$*,* $A_i \subseteq Y_i$ *for each* $i \in I$ *such that* $\forall (x, y) = (x^i, x_i, y) \in X \times Y \setminus N \times K \exists i \in I$ *and* $\exists \bar{u_i} \in B_i$*,* $\bar{v_i} \in A_i$ *satisfying* $\bar{u_i} \in D_i(x)$*,* $\bar{v_i} \in T_i(x)$ *and* $f_i(x, y, \bar{u_i}) \in -intC_i(x)$*.*

Then, there exists $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ *in* $X \times Y$ *such that for each* $i \in I$*,*

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \; f_i(\bar{x}, \bar{y}, z_i) \notin -intC_i(\bar{x}), \quad \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (W-SVQEP) is nonempty.

COROLLARY 3.2. *Let* I *be any index set. For each* $i \in I$*, let* Z_i *be a topological vector space,* E_i *and* F_i *be two locally convex Hausdorff topological vector spaces,* $X_i \subseteq E_i$ *and* $Y_i \subseteq F_i$ *be nonempty, closed and convex subsets, let* $D_i : X \to 2^{X_i}$ *and* $T_i : X \to 2^{Y_i}$ *be set-valued maps with nonempty convex values and open lower sections, the set* $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x)$ *and* $y_i \in T_i(x)\}$ *be closed in* $X \times Y$ *and* $f_i : X \times Y \times X_i \to Z_i$ *be a vector-valued function. Assume that the set-valued map* $D \times T = \left(\prod_{i \in I} D_i \times \prod_{i \in I} T_i\right) : X \times X \to 2^{X \times Y}$ *defined as* $(D \times T)(x, y) = \prod_{i \in I} D_i(x) \times \prod_{i \in I} T_i(y)$*,* $\forall (x, y) \in X \times X$*, is* Φ*-condensing and for each* $i \in I$*, the conditions* (i) *and* (ii) *of Corollary 3.2 hold. Then the solution set of the* (*W-SVQEP*) *is nonempty.*

Remark 3.10. (1) Let $C_i(x)$ be a proper closed and convex cone with apex at the origin and $intC_i(x) \neq \emptyset$ for each $i \in I$ and for all $x \in X$ and $P_i = \cap_{x \in X} C_i(x)$ for each $i \in I$. If $\forall i \in I$, $\forall x \in X, \varphi_i(x, x_i) \notin -intC_i(x)$ and $z_i \to \varphi_i(x, z_i)$ is natural P_i-quasifunction, then $\varphi_i(x, z_i)$ is weak type C_{i-x}-0-partially diagonally quasiconvex in the second argument. And the converse is not true in general. In Example 2.1, if we replace the set-valued map Ψ by a vector-valued function $\varphi_i : X \times X_i \to Z_i$ defined as

$$\varphi_i(x, z_i) = \langle \|x\| \|z_i\|_i p_i, z_i - x_i \rangle, \quad \forall (x, z_i) \in X \times X_i,$$

Then, it is easy to verify that $\varphi_i(x, z_i)$ is weak type C_{i-x}-0-partially diagonally quasiconvex in the second argument. However, $\varphi_i(x, z_i)$ is not natural P_i-quasifunction in z_i for some $x \in X$. In fact, choose $\hat{x} \in X$ such that $\langle p_i, \hat{x}_i \rangle > 0$, set $z_{i_1} = \frac{1}{2}\hat{x}_i$, $z_{i_2} = -\frac{1}{2}\hat{x}_i$. Then

$$\varphi_i(\hat{x}, z_{i_1}) = -\frac{1}{4}\|\hat{x}\|\|\hat{x}_i\|_i\langle p_i, \hat{x}_i\rangle \in -int P_i.$$
$$\varphi_i(\hat{x}, z_{i_2}) = -\frac{3}{4}\|\hat{x}\|\|\hat{x}_i\|_i\langle p_i, \hat{x}_i\rangle \in -int P_i.$$

But for $z_{i_0} = \frac{1}{2}(z_{i_1} + z_{i_2}) = 0$, we have

$$\varphi_i\left(\hat{x}, \frac{1}{2}(z_{i_1} + z_{i_2})\right) = \varphi_i(\hat{x}, z_{i_0}) = 0 \notin co(\varphi_i(\hat{x}, z_{i_1}), \quad \varphi_i(\hat{x}, z_{i_2})) - P_i$$

Hence, $\varphi_i(\hat{x}, z_i)$ is not natural P_i-quasifunction in z_i.

Let $Y = \{\bar{y}\}$. For each $i \in I$, let $T_i(x) = \{\bar{y}_i\}$, $\forall x \in X$, and $\varphi_i : X \times X_i \to Z_i$ be defined as $\varphi_i(x, z_i) = f_i(x, \bar{y}, z_i)$, $\forall (x, z_i) \in X \times X_i$. Then, by Corollary 3.1 and Corollary 3.2, respectively, we can obtain two new results which generalize Theorem 2 and Theorem 3 in [14] with more general convexity.

(2) For all $x \in X$, let $C_i(x) = C_i$ be a proper, closed and convex cone with apex at the origin and $intC_i \neq \emptyset$ for each $i \in I$. If $\forall i \in I$ and $\forall x \in X, \varphi_i(x, x_i) \notin -intC_i$ and $z_i \to \varphi_i(x, z_i)$ is C_i-quasifunction, then $\varphi_i(x, z_i)$ is weak type C_i-0-partially diagonally quasiconvex. And the converse is not true in general. In fact, it is easy to verify that φ_i in (1) is weak type C_i-0-partially diagonally quasiconvex but not C_i-quasifunction in the second argument. Let $Y = \{\bar{y}\}$. For each $i \in I$, for all $x \in X$, let $T_i(x) = \{\bar{y}_i\}$ and $D_i(x) = X_i$, let $\varphi_i : X \times X_i \to Z_i$ be defined as $\varphi_i(x, z_i) = f_i(x, \bar{y}, z_i)$, $\forall (x, z_i) \in X \times X_i$. Then, by Corollary 3.1, we can obtain a new results which generalize Theorem 2.2 of [19] with more general convexity.

(3) Theorems 3.1–3.5, Corollaries 3.1 and 3.2 extend and generalize Theorems 2 and 3 in [14] and Theorems 2.1 and 2.2 in [19] in several ways.

Remark 3.11. By Theorems 3.1–3.5, it is easy to get the existence results of solutions for other special cases of the four types of system of generalized vector equilibrium problems. And they are omitted here.

4. Applications

In this section, we present some existence of a solution for the (G-Debreu VEP).

THEOREM 4.1. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, E_i and F_i be two Hausdorff topological vector spaces, $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $intC_i(x) \neq \emptyset$ for each $x \in X$, $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the* *set* $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x) \quad \text{and} \quad y_i \in T_i(x)\}$ *be closed in* $X \times Y$ *and* ϕ_i *be a bifunction from* $X \times Y$ *into* Z_i. *For each* $i \in I$, *assume that*

(i) $M_i = Z_i \backslash (-intC_i) : X \to 2^{Z_i}$ *is upper semicontinuous;*
(ii) *For all* $x^i \in X^i$ *and* $y \in Y$, $z_i \mapsto \phi_i(x^i, y, z_i)$ *is natural* P_i*-quasifunction, where* $P_i = \cap_{x \in X} C_i(x)$;
(iii) ϕ_i *is continuous on* $X \times Y$;
(iv) *there exist nonempty and compact subsets* $N \subseteq X$ *and* $K \subseteq Y$ *and non-empty, compact and convex subsets* $B_i \subseteq X_i$, $A_i \subseteq Y_i$ *for each* $i \in I$ *such that* $\forall (x, y) = (x^i, x_i, y) \in X \times Y \backslash N \times K \exists i \in I$ *and* $\exists \bar{u_i} \in B_i$, $\bar{v_i} \in A_i$ *satisfying* $\bar{u_i} \in D_i(x)$, $\bar{v_i} \in T_i(x)$ *and* $\phi_i(x^i, y, \bar{u_i}) - \phi_i(x, y) \in -intC_i(x)$.

Then, there exists $(\bar{x}, \bar{y}) = (\bar{x}^i, \bar{x}_i, \bar{y}^i, \bar{y}_i)$ *in* $X \times Y$ *such that for each* $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}) \text{ and } \bar{y}_i \in T_i(\bar{x}) : \phi_i(\bar{x}^i, \bar{y}, z_i) - \phi_i(\bar{x}, \bar{y}) \notin -intC_i(\bar{x}), \forall z_i \in D_i(\bar{x}).$$

That is, the solution set of the (*G-Debreu VEP*) *is nonempty.*

Proof. For each $i \in I$, we define a trifunction $f_i : X \times Y \times X_i$ and a set-valued map $Q_i : X \times Y \to 2^{Z_i}$ as

$$f_i(x, y, u_i) = \phi_i(x^i, y, u_i) - \phi_i(x, y), \quad \forall (x, y, u_i) \in X \times Y \times X_i.$$
$$Q_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \in -intC_i(x)\}, \quad \forall (x, y) \in X \times Y.$$

Then $\forall i \in I$, $\forall (x, y) \in X \times Y$, $Q_i(x, y)$ is convex.

To prove it, let us fix arbitrary $i \in I$ and $(x, y) \in X \times Y$. Let $u_{i_1}, u_{i_2} \in Q_i(x, y)$ and $\lambda \in [0, 1]$, then we have

$$f_i(x, y, u_{i_j}) \in -intC_i(x), \qquad \text{for } j = 1, 2. \tag{2}$$

Since $\phi_i(x^i, y, .)$ is natural P_i quasifunction, by Remark 2 in [14], $\exists \alpha \in [0, 1]$ such that

$$\phi_i(x^i, y, \lambda u_{i_1} + (1 - \lambda) u_{i_2}) \in \alpha \phi_i(x^i, y, u_{i_1}) + (1 - \alpha) \phi_i(x^i, y, u_{i_2}) - P_i.$$

And hence

$$f_i(x, y, \lambda u_{i_1} + (1 - \lambda) u_{i_2}) \in \alpha f_i(x, y, u_{i_1}) + (1 - \alpha) f_i(x, y, u_{i_2}) - P_i. \tag{3}$$

From (2) and (3), we get

$$f_i(x, y, \lambda u_{i_1} + (1 - \lambda) u_{i_2}) \in -intC_i(x) - intC_i(x) - P_i \subseteq -intC_i(x).$$

Hence, $\lambda u_{i_1} + (1 - \lambda) u_{i_2} \in Q_i(x, y)$ and therefore $Q_i(x, y)$ is convex. Then we prove that for all $y \in Y$, $f_i(x, y, u_i)$ is weak type C_{i-x}-0-partially diagonally quasiconvex in the third argument. Otherwise, there is a point y in

Y, a finite subset $\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\}$ in X_i and a point $x=(x^i, x_i)$ in X with $x_i \in co\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\}$ such that $f_i(x, y, u_{i_j}) \in -intC_i(x)$ for all $j=1, 2, \ldots, n$. By the convexity of $Q_i(x, y)$, we have $x_i \in Q_i(x, y)$, that is, $f_i(x, y, x_i) = \varphi_i(x^i, y, x_i) - \varphi_i(x, y) = 0 \in -intC_i(x)$, which is absurd. Hence, for each $i \in I$, for all $y \in Y$, $f_i(x, y, u_i)$ is weak type C_{i-x}-0-partially diagonally quasiconvex in the third argument. It is easy to verify that the other conditions of Corollary 3.1 are satisfied. By Corollary 3.1, we know that the conclusion holds. □

By Corollary 3.2 and Lemma 2.2, we get

THEOREM 4.2. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, E_i and F_i be two locally convex Hausdorff topological vector spaces, $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty, closed and convex subsets, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $intC_i(x) \neq \emptyset$ for each $x \in X$, $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x) \quad and \quad y_i \in T_i(x)\}$ be closed in $X \times Y$ and $\varphi_i : X \times Y \to Z_i$ be a vector-valued function. Assume that the set-valued map $D \times T = (\prod_{i \in I} D_i \times \prod_{i \in I} T_i) : X \times X \to 2^{X \times Y}$ defined as $(D \times T)(x, y) = \prod_{i \in I} D_i(x) \times \prod_{i \in I} T_i(y)$, $\forall (x, y) \in X \times X$, is Φ-condensing and* (i), (ii) *and* (iii) *of Theorem* 4.1 *hold. Then, the solution set of the* (*G-Debreu VEP*) *is nonempty.*

Remark 4.1. Theorem 4.1 and Theorem 4.2 are new results even in scalar cases. Theorem 4.1 and Theorem 4.2, respectively, generalize Theorem 5 and Theorem 6 in [14].

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